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Control of a Three-Class Fluid Model with Routing

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Control of a Three-Class Fluid Model with Routing

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REPORT

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF ENGINEERING

THE UNIVERSITY OF TEXAS AT AUSTIN

December 2013

Dedicated to my husband Ziyu.

Control of a Three-Class Fluid Model with Routing

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This report studies the routing and scheduling control strategy of a three-class fluid model. A numerical approximation under the $c\mu$ scheduling policy is used. Analytical rules are provided to narrow down the optimal strategy under the policy. Numerical results and sensitivity analyses are presented to show how different control strategies perform given different parameters.

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Chapter 1

Introduction

Optimal scheduling for multiclass fluid networks has been a topic of great interest over the last 20 years. The fluid model is a powerful tool in dynamic scheduling of multiclass stochastic networks as has been demonstrated by many authors, with much of the work summarized by Chen and Yao [4]. Furthermore, Dai [7] showed the stability of fluid models implies stability of an associated queueing model. Avram et al. [1] derived the optimal control of a fluid model associated with open multiclass queueing networks and proposed a discretization approximation method to solve the problem as a linear program. Weiss [2] presented several problems of optimal control of re-entrant lines including minimizing the emptying time, the inventory at a target date, etc. Kulkarni [8] analyzed the case of a single buffer process with input and output rates depending on external environments.

None of the above research considers having control over the input flow. In this research report we study a single-server three-class fluid model with control over directing the input and output flow. This is similar to [5] where a two-class fluid model is presented. In that report, the sequencing rule is the $c\mu$ rule and we also employ that assumption here. Under the $c\mu$ rule priority is given to the according to the product of the fluid holding cost and processing rate, with higher values of the product given higher priority.In our report we first introduce the three-class model and the methodology to numerically solve an approximate model. We then attempt to narrow the possible optimal routing strategies. Although we could not obtain overarching analytical results, we derived several conditions that can be useful in excluding suboptimal strategies. In a case study we provide several cases that represents different control strategies. Finally in conclusion we summarize our results and propose some future research directions.

Chapter 2

Model Description

2.1 Model Setup

The three-fluid model discussed in this report is an extension of the two-fluid model. The report from [5] considered the system with two fluids, which we call type 1 and type 2. Similar to the two-fluid setting, the three-fluid model consists of three fluids with constant holding cost rates c_1, c_2 and c_3 per unit fluid per unit time, respectively. The amounts of fluid in the buffers are denoted by $Q_1(t)$, $Q_2(t)$ and $Q_3(t)$ for a fixed time t. The initial fluid inside buffer 1, 2 and 3 are denoted as $Q_1(0)$, $Q_2(0)$ and $Q_3(0)$. There is only one server processing the three fluids with a processing rate of μ_i for fluid i. As in the two-fluid model, we still assume the cost and processing rates follow the strong $c\mu$ rule: $c_1 > c_2 > c_3, \mu_1 > \mu_2 > \mu_3$. Note that aside from this condition, the server can process multiple fluids simultaneously assuming that the total workload does not exceed the capacity of the server. The incoming flow rate is λ where $\lambda < \mu_3$. So it is clear that the system is stable since each buffer has the capacity to handle the incoming flow by itself.

Denote T as the time that all buffers are drained. The total cost incurred is then:

$$\int_0^T [c_1 Q_1(t) + c_2 Q_2(t) + c_3 Q_3(t)] dt.$$

The objective in this model is to minimize the above cost. Figure 2.1 gives an illustration of the model.

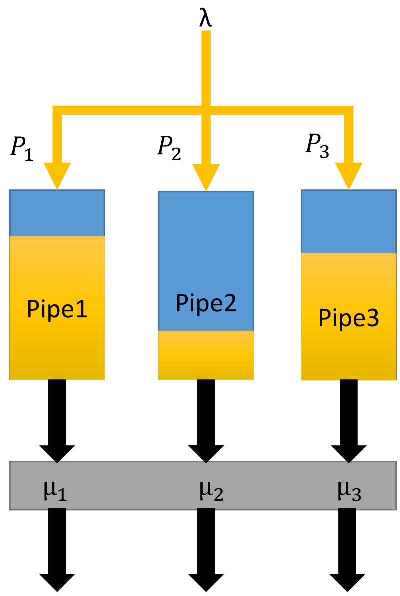

Figure 2.1: Three-Fluid Model

There are two general types of control in this system, routing and scheduling. We assume a priori that the $c\mu$ rule is used for scheduling. In that case, the only control decision is the routing policy. We define a routing policy by $p_i(t)$ which is the instantaneous proportion of class i fluid routed to the class i buffer. Of course, we have the constraint that $p_1(t) + p_2(t) + p_3(t) = 1$ for all $t \geq 0$.

Our methodology is to discretize the time interval, assume constant control over the subintervals, and then optimize these controls. As mentioned in the model description, we assume a strong $c\mu$ scenario, which is $c_1 > c_2 > c_3$ and $\mu_1 > \mu_2 > \mu_3$ are all constants. Under the $c\mu$ sequencing rule, we always devote the whole server capacity to process the type 1 fluid in buffer 1 as long as buffer 1 is not empty. For example, after buffer 1 is empty, we do not process the fluid in buffer 3 unless buffer 2 is also empty. (This does not imply the whole server capacity is used to process the fluid in buffer 2.) In particular, since we are fixing the sequencing strategy and then optimizing the routing strategy, there is no guarantee that this will produce a jointly optimal strategy.

2.2 Decomposition to Three Phases

It is clear that the amount of fluid inside the various buffers is an important factor in determining the overall cost of the draining process. If we direct all incoming fluid to buffer 1 just so the fluid can be processed faster, it will cost more since the cost rate in buffer 1 is higher. On the other hand, if we direct all the incoming fluid to buffer 2 or 3 since they have lower cost rates, the time before either of those buffers are empty is longer, which as a result incurs more cost. Therefore, there is no clear strategy of what to do with the incoming flow to minimize the overall cost. However, as we have illustrated before, according to the $c\mu$ rule we are at least able to decide to process type 1 fluid as long as buffer 1 is non-empty. Unlike the two-fluid model, here we decompose the draining process into three phases. We define phase 1 as the time from $t = 0$ until buffer 1 is empty, phase 2 as the time before buffer 2 is empty after phase 1, and phase 3 as the time before buffer 3 is empty after phase 2. Moreover, we only start processing fluids in buffer 2 after buffer 1 is empty, also we only start processing fluids in buffer 3 after buffer 2 is empty. In all of these phases, the decision we need to make is the proportion of fluid directed to buffer j in phase i at time t. We denote this instantaneous proportion as $P_{i,j}(t)$. In addition to the initial amounts of fluid in buffers 1, 2 and 3 denoted as Q_1^0 and Q_2^0, Q_3^0, we also denote the initial amount of fluid in buffer 2 and 3 at the beginning of phase 2 to be Q_2^1 and Q_3^1 whose values depend on the decisions we make in phase 1 and other parameters. Similarly we denote the initial amount of fluid in buffer 3 at the beginning of phase 3 to be Q_3^2 whose value depends on the decisions we make in the previous phases along with other parameters. However, since the analytical solution of $P_{i,j}(t)$ is likely intractable, instead we separate each phase into n periods, $n > 0$, where in each period of phase 1, a volume of $\frac{Q_1^0}{n}$ type 1 fluid is processed. Similarly in each period of phase 2 a volume of $\frac{Q_2^1}{n}$ type 2 fluid is processed. Finally in each period of phase 3 a volume of $\frac{Q_3^2}{n}$ type 3 fluid is processed. In this case our first set of decision variables becomes $P_{1,j}(k), P_{2,j}(k), P_{3,j}(k)$ in buffer j, where $k = 1, 2, 3, \ldots, 3n$, $j = 1, 2, 3$. Based on the definition of k it is easy to see that the time indices $k = 1, 2, 3, \ldots, n$ apply to phase 1, $k = n+1, n+2, n+3, \ldots, 2n$ apply to phase 2 and $k = 2n+1, 2n+2, 2n+3, \ldots, 3n$ apply to phase 3. Although phase number in the notation of proportion seems unnecessary, it is kept to indicate the phase. Finally defining t_k as the time of period k, the optimal strategy is defined via the quantities just defined.

Similar to the two-fluid model, the cost of one period k in phase i for a buffer j is

$$\frac{c_j t_k}{2}(2Q_j(k) + \lambda P_{i,j}(k) t_k)$$

where $Q_j(k)$ is the initial volume at beginning of period k in buffer j and which satisfies $Q_j(k+1) = Q_j(k) + \lambda P_{i,j}(k) t_k$ for $j > i$. For buffer i in phase i the cost in each period k is

$$\frac{c_i t_k Q_j^{i-1}}{2n}(2n + 1 - 2(k - (i-1)n)).$$

Here $t_k = \frac{Q_j^{i-1}}{n(\mu_i - \lambda P_{i,i}(k))}$. Thus we can formulate the total cost and solve an optimization problem to get the optimal fluid routing strategy. This way minimizing the total cost can be formulated as follows:

Index: $i = 1, 2, 3$

$j = 1, 2, 3$

$k_i \in N_i = \{(i-1)n + 1, (i-1)n + 2, \ldots, in\}$

Variables: $C_{ij}(k_i)$: The cost buffer j incurs in period k_i at phase i.

$P_{i,j}(k_i)$: The proportion of fluid that enters fluid j in period k_i at phase i.

t_{k_i} : The time length of period k_i (at phase i).

$Q_j(k_i)$: The amount of fluid in fluid j at beginning of period k_i.

Q_2^1, Q_3^1 : The amount of fluid in buffer 2 and 3 at the beginning of phase 2.

Q_3^2 : The amount of fluid in buffer 3 at the beginning of phase 3.

Data: c_j : The cost incurred in buffer j if one unit of fluid is held for one unit time.

Q_j^0 : The initial amount of fluid in buffer j.

λ : The incoming flow rate.

μ_i : The processing rate for fluid j (buffer j).

Formulation:

$$
\begin{aligned}
\min\ & \sum_{i=1}^{3}\sum_{j=i+1}^{3}\sum_{k_i\in N_i} C_{ij}(k_i) + \sum_{i=1}^{3}\sum_{k_i\in N_i} C_{ii}(k_i) \\
\text{s.t. } & C_{ii}(k_i) = \frac{c_i t_{k_i} Q_i^{i-1}}{2n}(2n+1-2(k_i-(i-1)n)) \text{ for } i=1,2,3 \\
& C_{ij}(k_i) = \frac{c_j t_{k_i}}{2}(2Q_j(k_i)+\lambda P_{i,j}(k_i)t_{k_i}) \text{ for } i=1,2,3 \text{ and } j>i \text{ and } k_i\in N_i \\
& t_{k_i} = \frac{Q_i^{i-1}}{n(\mu_i-\lambda P_{i,i}(k_i))} \text{ for } i=1,2,3 \text{ and } k_i\in N_i \\
& Q_j(k_i+1) = Q_j(k_i)+\lambda P_{i,j}(k_i)t_{k_i} \text{ for } i=1,2,3 \text{ and } j>i \text{ and } k_i\in N_i \\
& Q_j(1) = Q_j^0 \text{ for } j=1,2,3 \\
& Q_j^1 = Q_j(n+1) \text{ for } j=2,3 \\
& Q_3^2 = Q_3(2n+1) \\
& \sum_{j=1}^{3} P_{i,j}(k_i) = 1 \text{ for } i=1,2,3 \text{ and } k_i\in N_i \\
& P_{i,j}(k_i)\ge 0 \text{ for } \forall i,j \text{ and } \forall k_i\in N_i.
\end{aligned}
$$

In this formulation, the first two sets of constraints define the cost for each period. The third set of constraints defines the time length for each period. The fourth to seventh sets of constraints defines the fluid amount in each buffer period by period. Actually, the above formulation only contains the last two sets of constraints on the proportion, which are summation to one and the non-negativity constraints, if we plug in all other constraints into the objective function. The convexity of the problem is not guaranteed so it is likely that we can only derive a local optimum.

Chapter 3

Optimal Policy

In the two-fluid model, we know that in phase 1, the incoming fluid will either go entirely to buffer 1 or it will devote some proportion to buffer 2 and the rest to buffer 1. However, in contrast to the two-fluid model, in our three-fluid system, there can be more possibilities. Nevertheless, the following theory provides a restriction on the possible cases for the discrete problem. More specifically, when assuming a $c\mu$ sequencing rule, we separate the process into three phases based on the outflow strategy. The remaining question is how to allocate the inflow into buffers. In the following theorems we provide the conditions for the optimal inflow strategy under this outflow strategy.

Theorem 1. *Suppose the system parameters satisfy the strong $c\mu$ condition. If in period $k < n$ of phase 1 $P_{1,1}(k) = 1$, the fluid will not enter other buffers at period $k+1$ under an optimal inflow strategy, which means $P_{1,1}(k+1) = 1$.*

Proof. If in period $k < n$ of phase 1 $P_{1,1}(k) = 1$ and in period $k+1$ the fluid enters buffer 1, 2 and 3 are with proportion $P_{1,1}(k+1), P_{1,2}(k+1)$ and $P_{1,3}(k+1)$, where $P_{1,1}(k+1) < 1$ then it is sufficient to prove that this case can always be replaced by a better case with lower cost. For the case above $t_k = \frac{Q_1^0}{n(\mu_1 - \lambda)}$ and $t_{k+1} = \frac{Q_1^0}{n(\mu_1 - \lambda P_{1,1}(k+1))}$. The total cost for buffer 1 is:

$$\frac{c_1 Q_1^0}{2n}[t_k(2n-2k+1)+t_{k+1}(2n-2k-1)].$$

The total cost for buffer 2 is:

$$Q_2^0 c_2 t_k + \frac{t_{k+1}c_2}{2}(2Q_2^0 + \lambda P_{1,2}(k+1)t_{k+1}).$$

Similarly, the total cost for buffer 3 is:

$$Q_3^0 c_3 t_k + \frac{t_{k+1}c_3}{2}(2Q_3^0 + \lambda P_{1,3}(k+1)t_{k+1}).$$

Now consider another case which in period k the fluid enters buffer 1, 2 and 3 with proportion $P_{1,1}(k+1), P_{1,2}(k+1)$ and $P_{1,3}(k+1)$ while in period $k+1$ fluid fully enters buffer 1. Then the new time in period k is $t'_k = t_{k+1}$ and the new time in period $k+1$ is $t'_{k+1} = t_k$. The new total cost for buffer 1 is:

$$\frac{c_1 Q_1^0}{2n}[t_{k+1}(2n-2k+1)+t_k(2n-2k-1)].$$

The total cost for buffer 2 is:

$$c_2 t_k(Q_2^0 + \lambda P_{1,2}(k+1)t_{k+1}) + \frac{t_{k+1}c_2}{2}(2Q_2^0 + \lambda P_{1,2}(k+1)t_{k+1}).$$

And the total cost for buffer 3 is:

$$c_3 t_k(Q_3^0 + \lambda P_{1,3}(k+1)t_{k+1}) + \frac{t_{k+1}c_3}{2}(2Q_3^0 + \lambda P_{1,3}(k+1)t_{k+1}).$$

Some algebra shows that the total cost of the original case minus the total cost of the modified case is:

$$
\begin{aligned}
&\frac{(Q_1^0)^2}{n^2}c_1\Big(\frac{1}{\mu_1-\lambda}-\frac{1}{\mu_1-\lambda P_{1,1}(k+1)}\Big)-c_2\lambda P_{1,2}(k+1)t_kt_{k+1}-c_3\lambda P_{1,3}(k+1)t_kt_{k+1}\\
=&\frac{(Q_1^0)^2\lambda}{n^2(\mu_1-\lambda)(\mu_1-\lambda P_{1,1}(k+1))}[c_1(1-P_{1,1}(k+1))-c_2P_{1,2}(k+1)-c_3P_{1,3}(k+1)]\\
=&\frac{(Q_1^0)^2\lambda}{n^2(\mu_1-\lambda)(\mu_1-\lambda P_{1,1}(k+1))}[c_1(P_{1,2}(k+1)+P_{1,3}(k+1))-c_2P_{1,2}(k+1)-c_3\\
=&\frac{(Q_1^0)^2\lambda}{n^2(\mu_1-\lambda)(\mu_1-\lambda P_{1,1}(k+1))}[(c_1-c_2)P_{1,2}(k+1)+(c_1-c_3)P_{1,3}(k+1))]\\
\geq&0.
\end{aligned}
\tag{3.1}
$$

Equality holds in equation 3.1 only when $P_{1,1}(k+1)=1$, which establishes the result. □

It is very easy to derive the following corollaries via the same method above:

Corollary 1. *Suppose the system parameters satisfy the strong $c\mu$ condition. If in period $k<n$ of phase 1 $P_{1,2}(k)=1$, the fluid will not enter only buffer 2 and 3 at period $k+1$ under an optimal inflow strategy, which means $P_{1,1}(k+1)>0$.*

Corollary 2. *Suppose the system parameters satisfy the strong $c\mu$ condition. If in period $k<n$ of phase 1 fluid enters buffer i_1 and i_2 where $i_1<i_2$ with proportion $P_{1,i_1}(k)$ and $P_{1,i_2}(k)$ and $P_{1,i_1}(k)+P_{1,i_2}(k)=1$, and the fluid enter exactly same buffers in period $k+1$ with proportion $P_{1,i_1}(k+1)$ and $P_{1,i_2}(k+1)$ and $P_{1,i_1}(k+1)+P_{1,i_2}(k+1)=1$, then it is always true that $P_{1,i_1}(k)\leq P_{1,i_1}(k+1)$ and $P_{1,i_2}(k)\geq P_{1,i_2}(k+1)$ under an optimal inflow strategy.*

Corollary 3. *Suppose the system parameters satisfy the strong $c\mu$ condition. If in period $k<n$ of phase 1 $P_{1,1}(k)+P_{1,2}(k)>0$, then in period $k+1$ we have $P_{1,3}(k+1)<1$ under an optimal inflow strategy.*

All the three corollaries can be proven by straightforward interchange arguments similar to Theorem 1.

From the results above, it can be seen that in phase 1, the order that fluid enters the buffer is always from buffer 3 to 2 to 1 and cannot be reversed. Even if in some periods fluid enters more than one buffer, the proportion of fluid entering a lower order buffer (buffer number large) will gradually decrease and move to a higher order buffer (buffer number small).

Theorem 2. *Under the optimal routing strategy, in phase 3,* $P_{3,1}(k) = 1, 2n < k \leq 3n$.

Proof. The amount of fluid in buffer 3 is already fixed and both buffer 1 and 2 are empty at the beginning of phase 3. Since $\mu_1 > \lambda$ and $\mu_2 > \lambda$ there will be no cost in these two buffers as fluid will not accumulate. Thus we only need to choose $P_{3,j}(k)$ so that the draining rate is maximized, thus minimizing the holding cost. As $\mu_1 > \mu_2 > \mu_3$, it is clear that the draining rate in period k in buffer 3 is:

$$\begin{aligned}
&\mu_3(1 - \frac{\lambda P_{3,1}(k)}{\mu_1} - \frac{\lambda P_{3,2}(k)}{\mu_2}) - \lambda(1 - P_{3,1}(k) - P_{3,2}(k)) \\
=&\mu_3 - \lambda + \lambda[P_{3,1}(k)(1 - \frac{\mu_3}{\mu_1}) + P_{3,2}(k)(1 - \frac{\mu_3}{\mu_2}) \\
\leq&\mu_3 - \lambda + \lambda[P_{3,1}(k)(1 - \frac{\mu_3}{\mu_1}) + P_{3,2}(k)(1 - \frac{\mu_3}{\mu_1}) \qquad (3.2)\\
\leq&\mu_3 - \lambda + \lambda(P_{3,1}(k) + P_{3,2}(k))(1 - \frac{\mu_3}{\mu_1}) \\
\leq&\mu_3 - \lambda + \lambda(1 - \frac{\mu_3}{\mu_1}).
\end{aligned}$$

The last expression is a constant and equality holds only when $P_{3,2}(k) = 0$ and $P_{3,1}(k) = 1$. It is worth mentioning here that since $\mu_2 < \mu_1$, the first inequality only holds when $P_{3,2}(k) = 0$. The above is true for all k, which means that fluid only enters buffer 1 during entire phase 3. □

Theorem 3. *Under the optimal routing strategy, in phase 2, $P_{2,2}(k) = 0, \forall n < k \leq 2n$.*

Proof. In phase 2, the fluid entering buffer 1 will not accumulate so there is no cost in that buffer. Now fix $P_{2,3}(k)$ for all k, thus the proportion of fluid entering buffer 3 in each period of phase 2 is fixed. In this way the total cost incurred in buffer 2 and 3 during phase 2 is minimized if the draining rate in each period of phase 2 is maximized. (The time for each period is minimized.)

On the other hand the total cost incurred in buffer 3 in phase 3 is also minimized if the draining rate in each period of phase 2 is maximized since the total amount of fluid in buffer 3 at the beginning of phase 3 is minimized. Note that in general we do not need such a strong condition. The only objective is to minimize the cost of both buffer 2 and 3 in phase 2 plus the cost of buffer 3 in phase 3. What we are trying to show is a much stronger condition.

When $P_{2,3}(k)$ is fixed the draining rate in buffer 2 at period k is:

$$\begin{aligned}
&\mu_2(1 - \frac{\lambda P_{2,1}(k)}{\mu_1} - \frac{\lambda P_{2,3}(k)}{\mu_2}) - \lambda P_{2,2}(k) \\
=&\mu_2(1 - \frac{\lambda(1 - P_{2,3}(k))}{\mu_1} + \frac{\lambda P_{2,2}(k)}{\mu_1} - \frac{\lambda P_{2,3}(k)}{\mu_2}) - \lambda P_{2,2}(k) \\
=&\mu_2(1 - \frac{\lambda(1 - P_{2,3}(k))}{\mu_1} - \frac{\lambda P_{2,3}(k)}{\mu_2}) + \lambda P_{2,2}(k)\frac{\mu_2}{\mu_1} - \lambda P_{2,2}(k) \qquad (3.3)\\
=&\mu_2(1 - \frac{\lambda(1 - P_{2,3}(k))}{\mu_1} - \frac{\lambda P_{2,3}(k)}{\mu_2}) - \lambda P_{2,2}(k)(1 - \frac{\mu_2}{\mu_1}) \\
\leq&\mu_2(1 - \frac{\lambda(1 - P_{2,3}(k))}{\mu_1} - \frac{\lambda P_{2,3}(k)}{\mu_2}).
\end{aligned}$$

The inequality holds since $\mu_2 < \mu_1$. Equality holds when $P_{2,2}(k) = 0$. Since it is true for all k, and $P_{2,3}(k)$ is arbitrary, this implies that we should not direct any fluid into buffer 2 in any period no matter how we choose $P_{2,3}(k)$.

□

From Theorem 3 it can be seen that in phase 2 fluid cannot be directed to buffer 2, which means $P_{2,2}(k) = 0$. Now we have a similar corollary to Corollary 2.

Corollary 4. *Assume the strong $c\mu$ is true. If in period $k \in (n, 2n)$ of phase 2 fluid enters buffer 1 and 3 with proportion $P_{1,1}(k)$ and $P_{1,3}(k)$ and in period $k+1$ with proportion $P_{1,1}(k+1)$ and $P_{1,3}(k+1)$, it is always true that $P_{1,1}(k) \leq P_{1,1}(k+1)$ and $P_{1,3}(k) \geq P_{1,3}(k+1)$ under an optimal strategy.*

The proof is very similar to the proof of Theorem 1. Above all, it can be seen that in phase 1, fluid can enter all three buffers, while in phase 2 fluid can only enter buffer 1 and 3 and in phase 3 fluid can only enter buffer 1.

Chapter 4

Case Study

4.1 Sample Cases

As we mentioned in Chapter 2, in the three-fluid system there can be many more cases in the optimal scheduling strategy of the incoming fluid than in a two-fluid system. The theorems above eliminate many cases but there are still lots of cases remaining. In this section, we use MINOS [10] and some other nonlinear programming solvers in GAMS [9] and numerically solve the discretized problem in which we divide each phase into 200 periods. Each of the solvers gives a very similar solution, which provides some numerical guidance of the optimality (local optimality). For simplicity we now fix $\lambda = 5$, $Q_1^0 = 100$, $Q_2^0 = 100$, and $Q_3^0 = 100$. By changing the cost and processing rates we examine the following five cases:

1. Case 1 Routing Sequence:

   Phase 1: For some $k_1 > 0$, $P_{1,3}(k_1') = 1$ where $k_1' \leq k_1$. For some $k_2 \geq k_1$, $P_{1,3}(k_2') \leq P_{1,3}(k_2' + 1)$ and $P_{1,3}(k_2') + P_{1,1}(k_2') = 1$ where $k_1 < k_2' \leq k_2$. $P_{1,1}(k_3') = 1$ where $k_2 < k_3' \leq n$.

   Phase 2: $P_{2,1}(k) = 1$, where $k = n+1, n+2, n+3, \ldots, 2n$

   Phase 3: $P_{3,1}(k) = 1$, where $k = 2n+1, 2n+2, 2n+3, \ldots, 3n$

   Cost: $c_1 = 20, c_2 = 18, c_3 = 17$

   Processing Rate: $\mu_1 = 20, \mu_2 = 19, \mu_3 = 18$

2. Case 2 Routing Sequence:

   Phase 1: $P_{1,1}(k) = 1$, where $k = 1, 2, 3, \dots, n$

   Phase 2: $P_{2,1}(k) = 1$, where $k = n + 1, n + 2, n + 3, \dots, 2n$

   Phase 3: $P_{3,1}(k) = 1$, where $k = 2n + 1, 2n + 2, 2n + 3, \dots, 3n$

   Cost: $c_1 = 20, c_2 = 18, c_3 = 17$

   Processing Rate: $\mu_1 = 20, \mu_2 = 10, \mu_3 = 6$

3. Case 3 Routing Sequence:

   Phase 1: For some $k_1 > 0$, $P_{1,2}(k'_1) = 1$ where $k'_1 \leq k_1$. For some $k_2 \geq k_1$, $P_{1,2}(k'_2) \leq P_{1,2}(k'_2 + 1)$ and $P_{1,2}(k'_2) + P_{1,1}(k'_2) = 1$ where $k_1 < k'_2 \leq k_2$. $P_{1,1}(k'_3) = 1$ where $k_2 < k'_3 \leq n$.

   Phase 2: $P_{2,1}(k) = 1$, where $k = n + 1, n + 2, n + 3, \dots, 2n$

   Phase 3: $P_{3,1}(k) = 1$, where $k = 2n + 1, 2n + 2, 2n + 3, \dots, 3n$

   Cost: $c_1 = 20, c_2 = 18, c_3 = 17$

   Processing Rate: $\mu_1 = 20, \mu_2 = 19, \mu_3 = 5$

4. Case 4 Routing Sequence:

   Phase 1: $P_{1,3}(k) = 1$, where $k = 1, 2, 3, \dots, n$

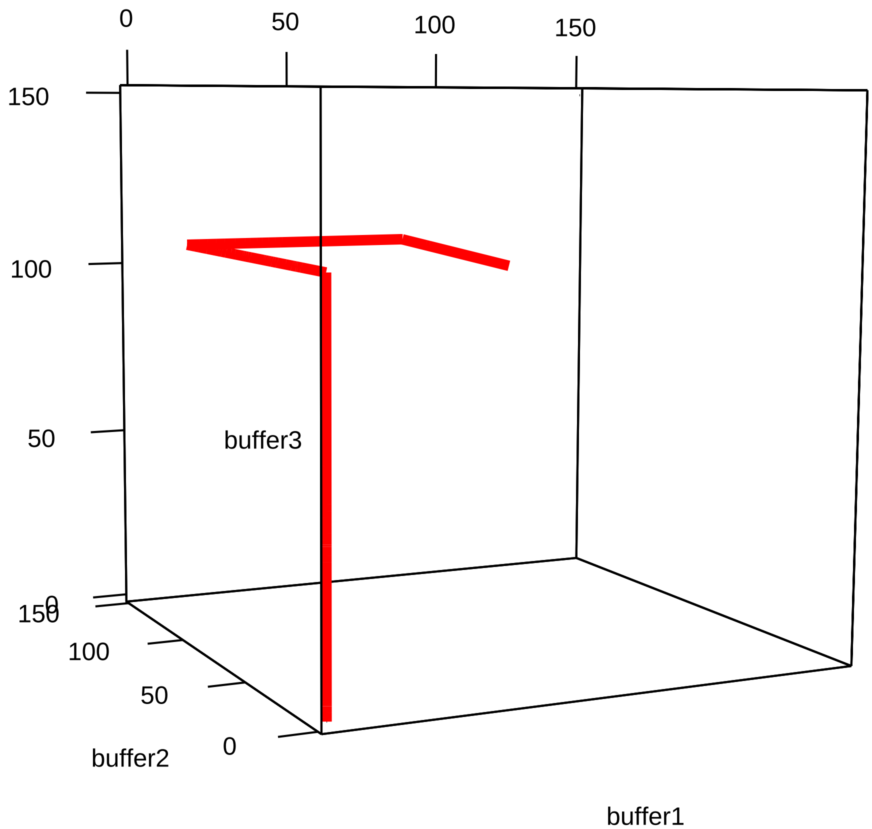

Figure 4.1: Case 1

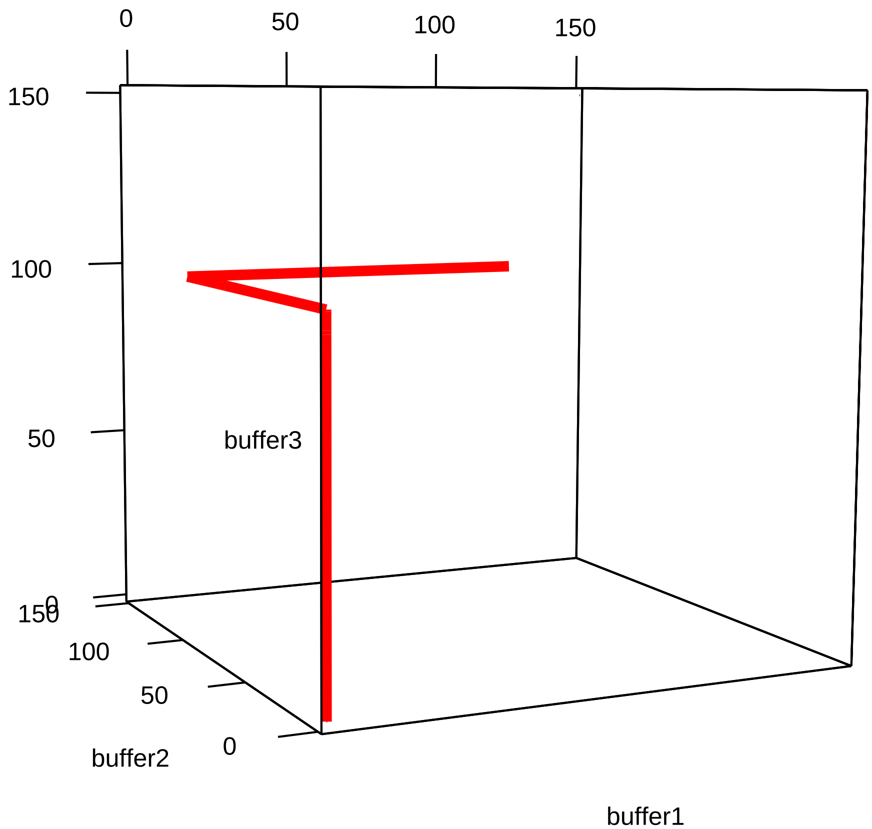

Figure 4.2: Case 2

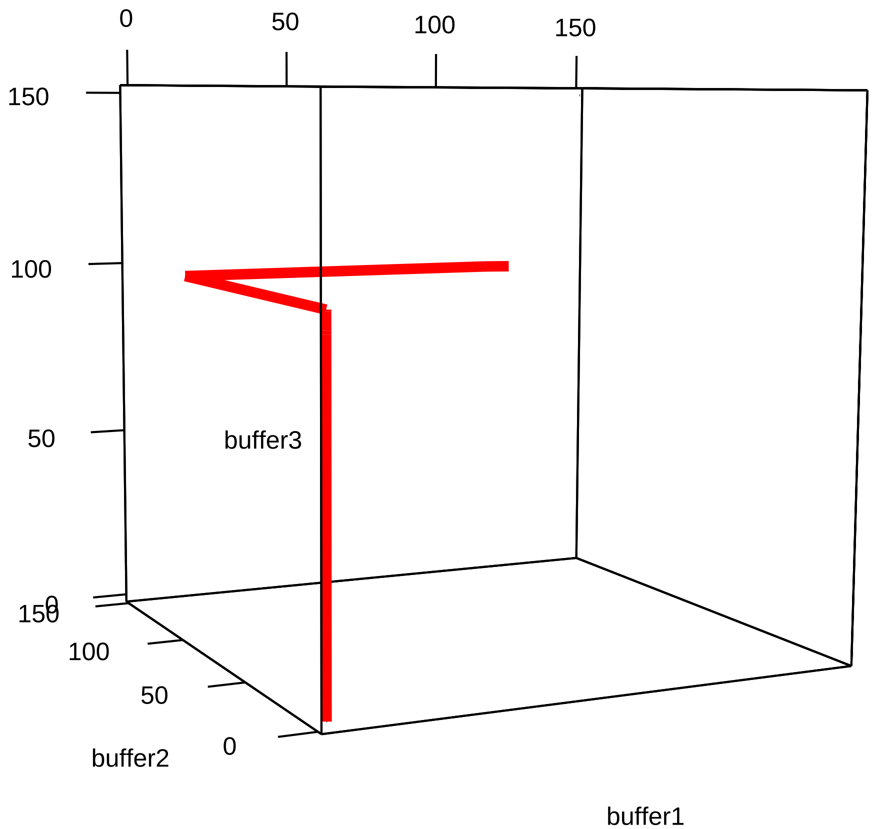

Figure 4.3: Case 3

Phase 2: For some $k_1 > n$, $P_{2,3}(k_1') = 1$ where $k_1' \leq k_1$. For some $k_2 \geq k_1$, $P_{2,3}(k_2') \leq P_{2,3}(k_2' + 1)$ and $P_{2,3}(k_2') + P_{2,1}(k_2') = 1$ where $k_1 < k_2' \leq k_2$. $P_{2,1}(k_3') = 1$ where $k_2 < k_3' \leq 2n$.

Phase 3: $P_{3,1}(k) = 1$, where $k = 2n+1, 2n+2, 2n+3, \ldots, 3n$

Cost: $c_1 = 20, c_2 = 18, c_3 = 5$

Processing Rate: $\mu_1 = 20, \mu_2 = 19, \mu_3 = 18$

5. Case 5 Routing Sequence:

   Phase 1: $P_{1,3}(k) = 1$, where $k = 1, 2, 3, \ldots, n$

   Phase 2: $P_{2,3}(k) = 1$, where $k = n+1, n+2, n+3, \ldots, 2n$

   Phase 3: $P_{3,1}(k) = 1$, where $k = 2n+1, 2n+2, 2n+3, \ldots, 3n$

   Cost: $c_1 = 20, c_2 = 18, c_3 = 0.001$

   Processing Rate: $\mu_1 = 20, \mu_2 = 19, \mu_3 = 18$

Figures 4.1 to 4.5 show the amount of fluid in buffer 1, 2 and 3 during the draining process. The red line represents the amount of fluid inside each buffer at any point of the draining process. As we can see from the five plots, without changing the incoming flow rate and the initial fluid in each buffer, and just by changing the cost and processing rate we are able to conclude that in all of the examples all incoming fluid is directed to buffer 1 in phase 3 which is consistent with Theorem 2. Furthermore, we can also see from the five examples that no incoming fluid is directed to buffer 2 in phase 2. This observation is consistent with

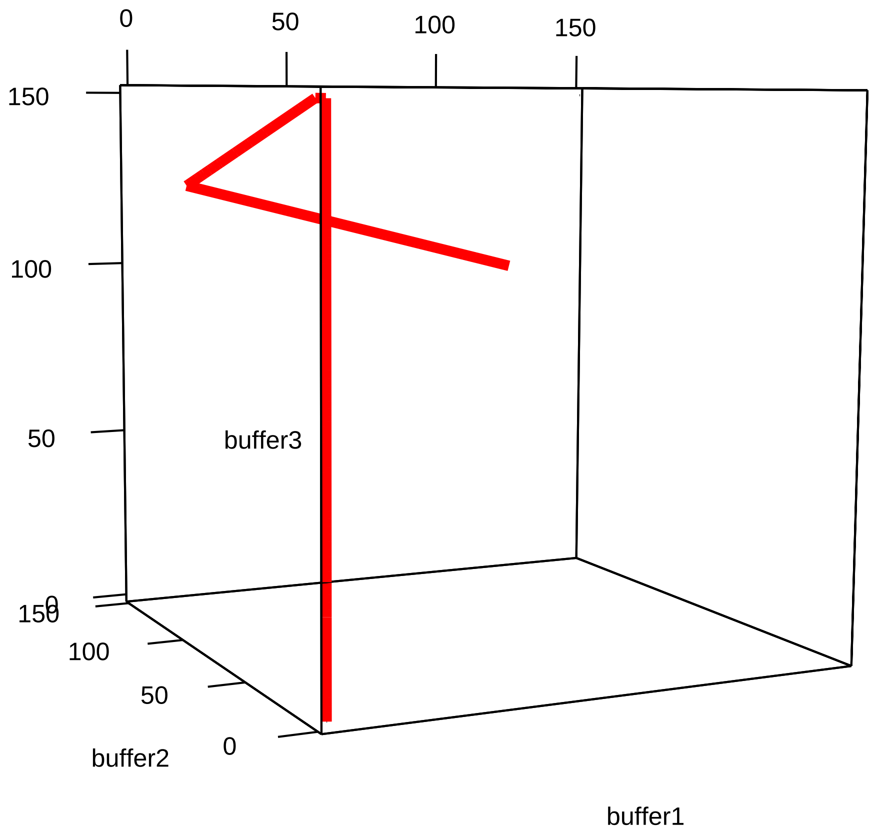

Figure 4.4: Case 4

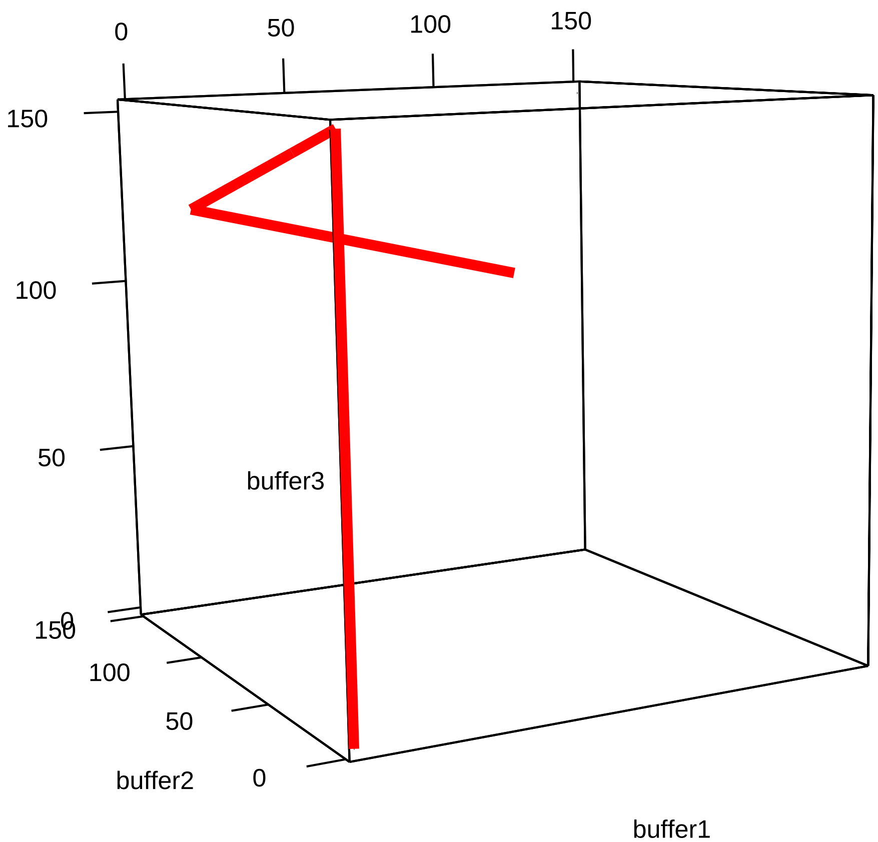

Figure 4.5: Case 5

Theorem 3. However, apart from these two qualitative characteristics implied by the theory we can also see that in all the cases the optimal solution seems to be following the same pattern. As a result, in each individual phase, once the incoming fluid starts to enter buffer 1 it continues to be directed to buffer 1 until the end of this phase.

4.2 Sensitivity Analysis

Besides the five cases studied above, we are also interested in the effects of an individual parameter on the optimal solution. Hence, we now undertake a limited sensitivity analysis. The analysis is done by fixing all but one parameter. In the sensitivity analysis for parameter μ_3, we see in Figure 4.6 that the optimal cost is decreasing in μ_3, as this parameter is varied from 6 to 18. Not only does the optimal cost change as μ_3 decreases, but the optimal policy also changes: When μ_3 is increased to 18, the optimal routing sequence of the incoming fluid changes from going first to buffer 2 then to buffer 1 (case 3), to going first to buffer 3 then to buffer 1 (case 1). Figure 4.7, illustrates the two different optimal routing sequences as μ_3 increases. This change of optimal strategy makes sense since as μ_3 increases it takes less time to process type 3 fluid and thus the optimal allocation of the incoming fluid tends to prioritize directing fluid to buffer 3 instead of buffer 2.

In the sensitivity analysis of parameter μ_2, Figure 4.9 suggests that as μ_2 increases the optimal cost decreases accordingly. As before, the optimal strategy also changes when $\mu_2 = 19$. In this case the optimal allocation of the incoming fluid changes from directing all incoming fluid to buffer 1 (case 2), to first directing fluid to buffer 2 then to buffer 1 (case 3). This again makes sense since as μ_2 increases the time needed to process type 2 fluid decreases, and the optimal allocation of the incoming fluid directs fluid to buffer 2.

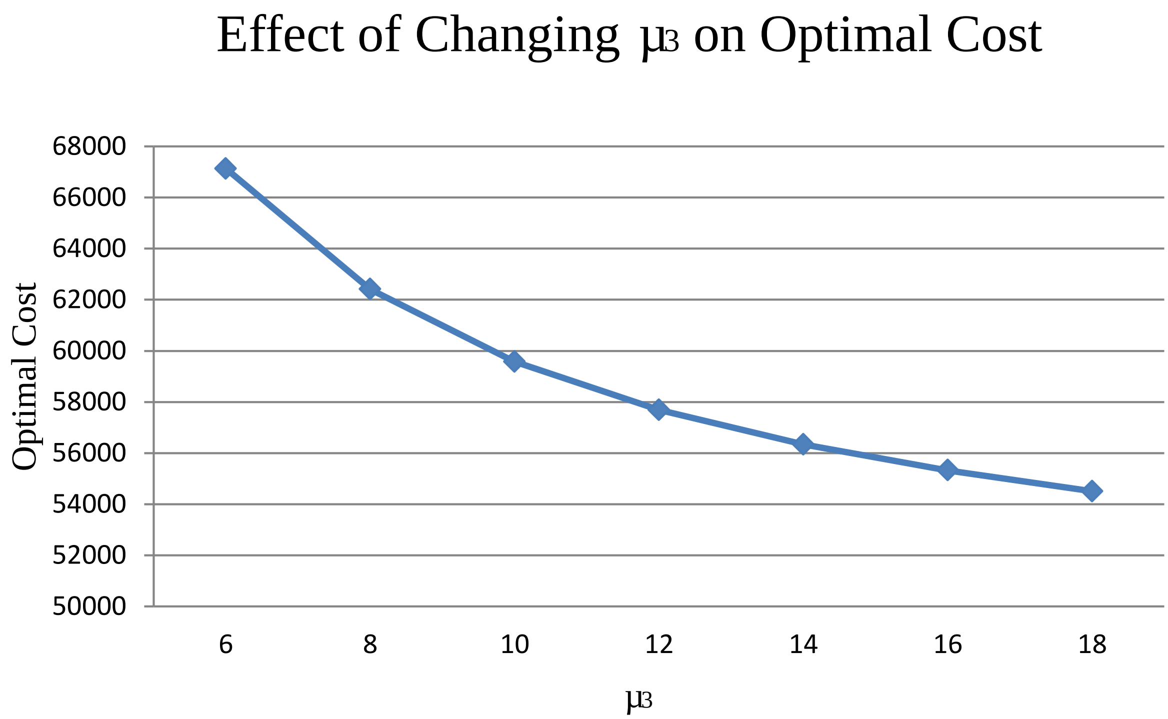

Figure 4.6: Sensitivity of Cost to μ_3

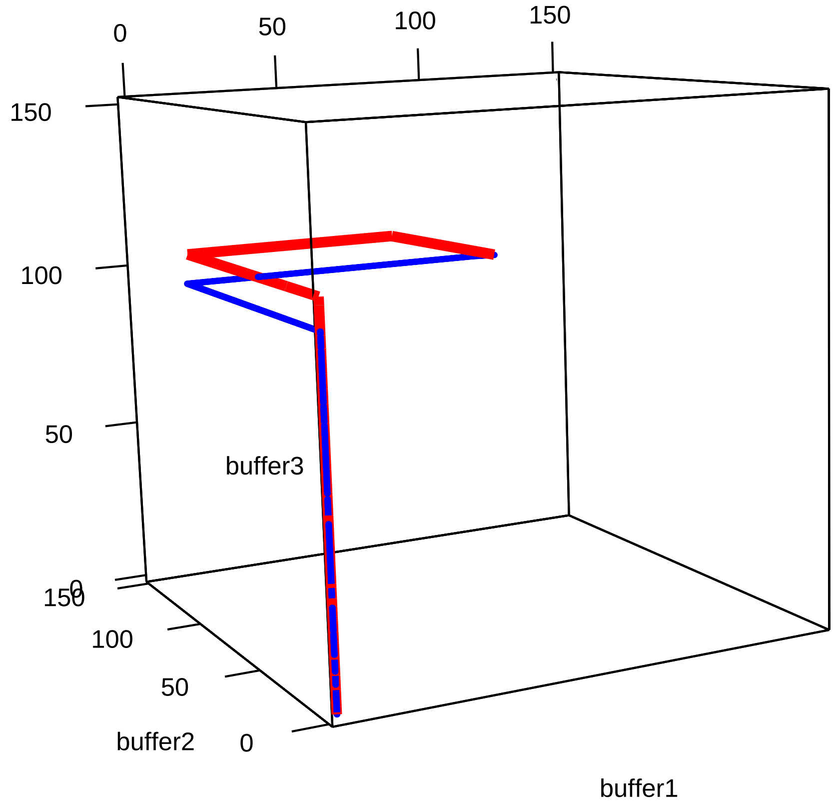

Figure 4.7: Sensitivity of Strategy to μ_3

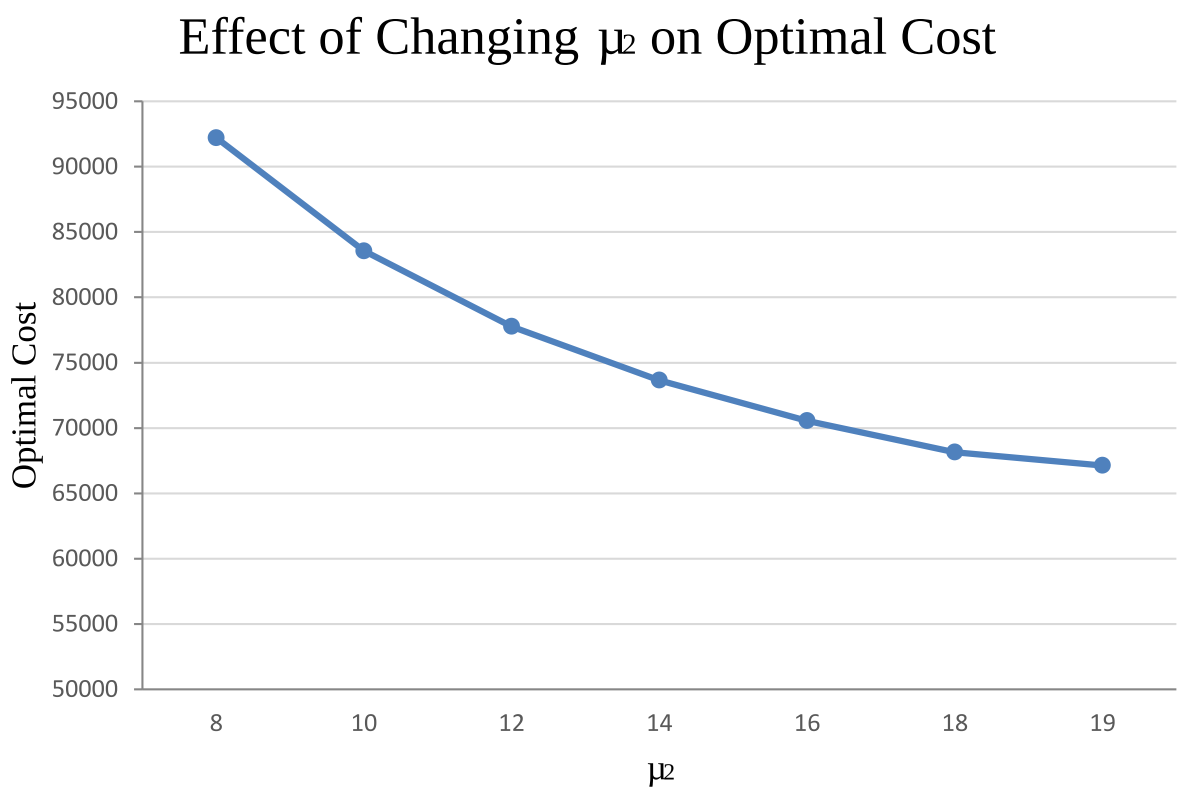

Figure 4.8: Sensitivity of Cost to μ_2

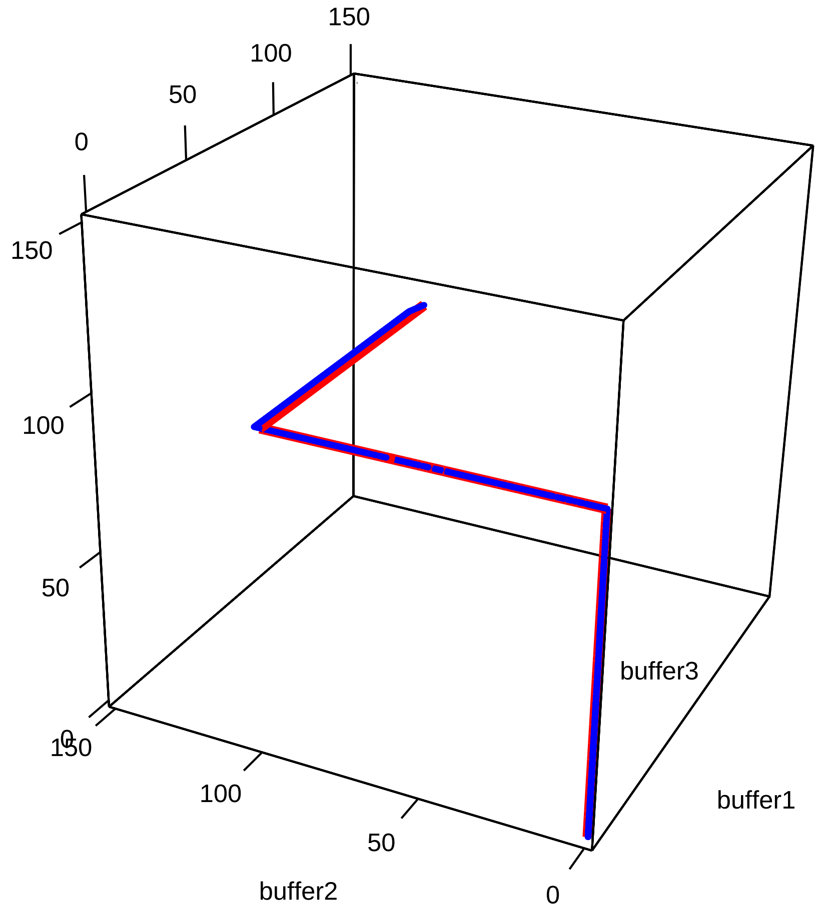

Figure 4.9: Sensitivity of Strategy to μ_2

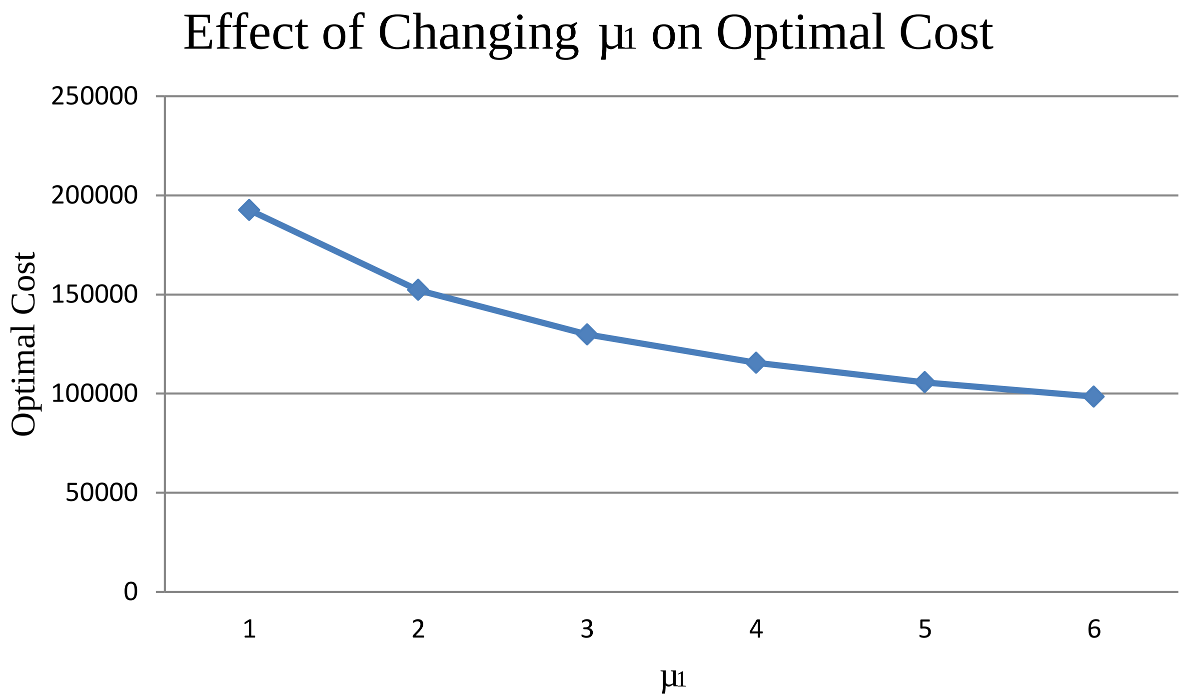

Figure 4.10: Sensitivity of Cost to μ_1

It is interesting that we are able to change the optimal strategy by changing only a single parameter. Figure 4.9 show the amount of fluid inside each buffer at each point in time under the two different cases.

In the sensitivity analysis with respect to parameter μ_1, the optimal policy does not change qualitatively with a change in μ_1 only. However, we can still see from Figure 4.10 that as μ_1 increases the corresponding optimal total cost decreases. This makes sense, since the optimal strategy in this case is to direct all incoming fluid to buffer 1. Increasing the processing rate for buffer 1 significantly reduces the time needed to process the fluids in each buffer, hence reducing the total cost.

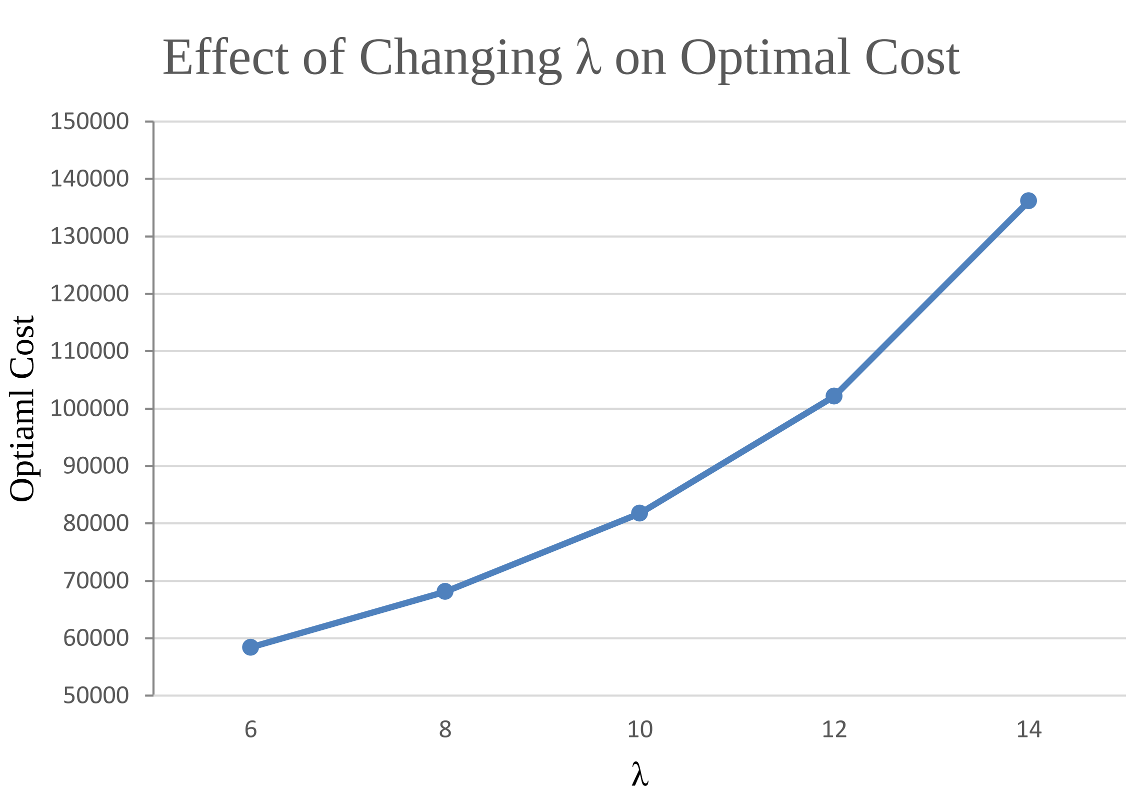

Figure 4.11: Sensitivity of Cost to λ

Figure 4.11 shows the relationship between the input rate and the optimal total cost. As we can see from the plot, as λ increases the optimal total cost increases as well. This is intuitive since the higher the input rate, the heavier the burden on the system to process incoming fluids, in addition to the initial fluid.

4.3 An Extra Case

Based on all the previous findings, we notice that in phase 3 there can be only one strategy, which is to direct all incoming fluid to buffer 3. In phase 2 the incoming fluid can be directed to two different buffers, buffer 1 and 3, and depending on the parameters, all fluid can be directed to buffer 1 or buffer 3, or first to buffer 3 then to buffer 1. In phase 1, we have only observed cases in which fluid is directed to two different buffers. A remaining question is whether it is possible that in this phase the fluid can be directed to all buffers under an optimal strategy. If we look into our sensitivity analysis on μ_3 and μ_2 more closely we can see that we are able to change the optimal strategy with the change of a single parameter. Under an approximately defined notion of continuity for sets of functions, if we are able to change the parameter by a sufficiently small amount the optimal cost based on the optimal strategy should also be changed by only a little although the strategy itself can be very different. So another sensitivity analysis was performed specifically to try to find the critical point for the change of the optimal strategy and also find a case where all three buffers are used in the optimal allocation of the incoming fluid in phase 1. This goal is achieved in the following case:

Case 6:

Phase 1: For some $k_1 > 0$, $P_{1,3}(k_1') = 1$ where $k_1' \leq k_1$. For some $k_2 \geq k_1$, $P_{1,3}(k_2') \leq$

$P_{1,3}(k_2'+1)$ and $P_{1,3}(k_2')+P_{1,2}(k_2')=1$ where $k_1<k_2'\leq k_2$. For some $k_3>k_2$, $P_{1,2}(k_3')=1$ where $k_2<k_3'\leq k_3$. For some $k_4\geq k_3$, $P_{1,2}(k_4')\leq P_{1,2}(k_4'+1)$ and $P_{1,2}(k_4')+P_{1,1}(k_4')=1$ where $k_3<k_4'\leq k_4$. $P_{1,1}(k_5')=1$ where $k_4<k_5'\leq n$.

Phase 2: $P_{2,1}(k)=1$, where $k=n+1,n+2,n+3,\ldots,2n$

Phase 3: $P_{3,1}(k)=1$, where $k=2n+1,2n+2,2n+3,\ldots,3n$

Cost: $c_1=20,c_2=18,c_3=17$

Processing Rate: $\mu_1=20,\mu_2=19,\mu_3=17.11$

This disapproves a conjecture we had made that the optimal strategy only ever directs fluid to two of the buffers in phase 1.

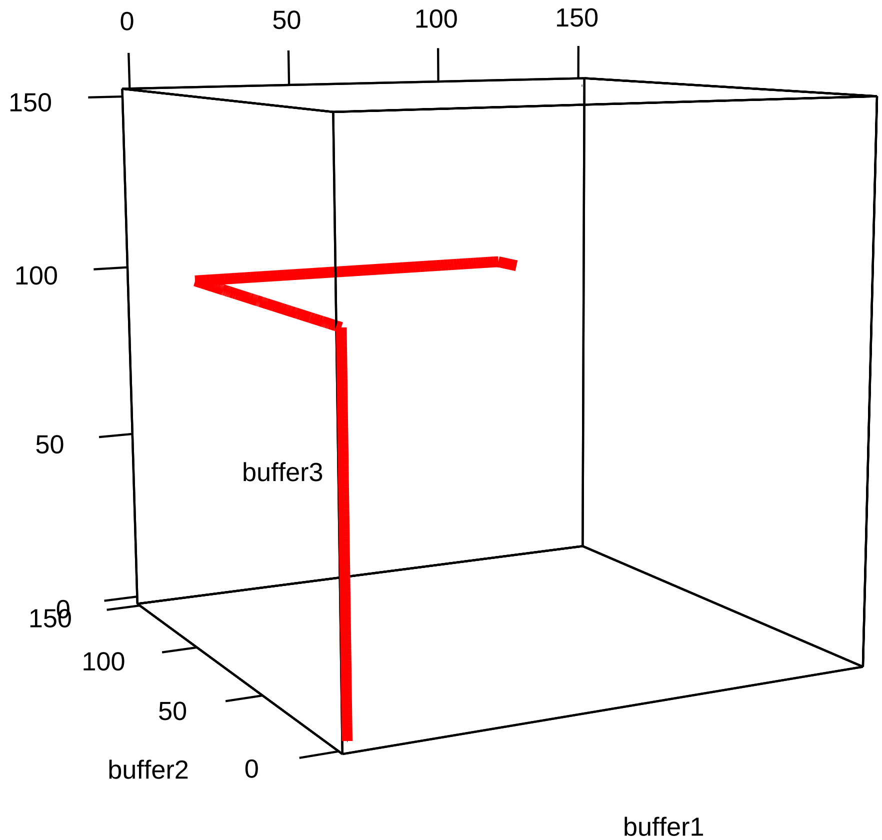

Figure 4.12: Case 6

Chapter 5

Conclusion

In queueing theory, fluid models are often used to approximate discrete queueing models. The fluid model we have described in this report is a system consisting of three fluids each with its own holding cost and processing rate. The incoming flow rate is set purposely lower than the smallest processing rate of the pipes to ensure a stable system. Through the optimal solution of this particular fluid system we wish to approximate the optimal policy in a related three-class queueing model. In the fluid model analyzed in this report, we focus on the optimal fluid routing policy.

In order to derive an approximately optimal solution, we decomposed the draining process into three qualitative phases. We divided each phase into n different periods, and each period is defined as the time to process the same amount of fluid. Finally we solve this nonlinear programming problem using the solver such as MINOS in GAMS.

In addition to proving theoretical results, we have done a case study to further illustrate possible optimal solution paths. Six different paths have been found and one of which even uses all three buffers during the draining process. However, the cases examined are nowhere near exhaustive. There still remain some certain cases of the routing sequence we can neither exclude as suboptimal nor provide an example that is optimal. Above all, we are able to formulate the three-fluid model into a discretized nonlinear programming problem, and solve the problem which provides some guidance of the optimal (local optimal) solution for different choice of parameters. Although we have developed many theorems in describing the optimal solution for the discretized problem, we are still not able to develop an analytical solution to this three-fluid model. Developing a complete analytical solution remains for future work.

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Vita

Meimeizi Zhu was born in Shanghai, China on August 28th. 1988, the daughter of Linsheng Zhu and Meijuan Wang. She received the Bachelor of Science degree in Statistics from the University of California Los Angeles. She applied to the University of Texas at Austin for enrollment in their Operations Research and Industrial Engineering program. She was accepted and started graduate studies in August, 2011.

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[†] LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.